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Limits of Cartier divisors

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ABSTRACT

Consider a one-parameter family of algebraic varieties degenerating to a reducible one. Our main result is a formula for the fundamental cycle of the limit subscheme of any family of effective Cartier divisors. The formula expresses this cycle as a sum of Cartier divisors, not necessarily effective, of the components of the limit variety.

1. Introduction

Consider a local one-parameter family of Noetherian schemes. More precisely, let $f : X \to S$ be a flat map of Noetherian schemes, where S stands for the spectrum of a discrete valuation ring. Let s and η denote the special and generic points of S; put $X_s := f^{-1}(s)$ and $X_\eta := f^{-1}(\eta)$. Assume that X_s is of pure dimension and has no embedded components.

Let D be an effective Cartier divisor of X. View it as a subscheme of X, and let $\lim D$ be the schematic boundary of $D \cap X_\eta$. Then $\lim D \subseteq D \cap X_s$. Equality does not necessarily hold, as D may contain components of X_s in its support.

This note presents a formula for the fundamental cycle $[\lim D]$ of $\lim D$ in terms of Cartier divisors of the components of X_s; see Theorem 4.1. The idea used in its proof is that, even though D may not restrict to an effective Cartier divisor of a given component of X_s, a suitable modification of D may. Suitable modifications may not exist. They do when X_s is reduced, a consequence of Proposition 4.3. At any rate, when they exist, a formula for $[\lim D]$ is derived by keeping track of the modifications and their restrictions to the components of X_s.

The idea used in the proof of the main theorem is reminiscent of that behind the definition of limit linear series, as explained in [1]. And, in fact, the main application of the theorem so far is in computing limits of ramification points of families of linear systems. The theorem is perfectly adapted for dealing with the case of plane curves, the study of which will be done in [3]. Example 5.3 is given to show, in a very simple situation, how the theorem will be applied there.

A rough layout of the paper is as follows. In Section 2 we define modifications, and present the main technical lemmas that will allow us to keep track of them later on. Section 3 is devoted to defining cycles and limit cycles, and proving a few of their fundamental properties, among them Proposition 3.2, stating that taking the fundamental cycle of the limit is additive for Cartier divisors. Section 4 is the heart of the notes, containing the main result, Theorem 4.1, and the auxiliary Proposition 4.3, giving conditions for when the theorem may be applied. Finally, in Section 5 we present examples to show how Theorem 4.1 can be applied.

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2. Modifications

2.1. Setup

Throughout the paper, S will stand for the spectrum of a discrete valuation ring, s for its closed point and η for its generic point. Also, π will denote a local parameter of S at s.

Throughout the paper, $f : X \to S$ will stand for a map from a Noetherian scheme X. Set $X_s := f^{-1}(s)$ and $X_\eta := f^{-1}(\eta)$. We call X_s the *special fiber* and X_η the *generic fiber* of f. We will always assume that X_s has no embedded components. Denote by $C_1, \dots, C_n$ the subschemes of X_s defined by the primary ideal sheaves of 0 in $\mathcal{O}_{X_s}$, and by $\xi_1, \dots, \xi_n$ their generic points. We say that $C_1, \dots, C_n$ are the *irreducible primary subschemes* of X_s.

A union of irreducible primary subschemes of X_s, defined by the intersection of the corresponding sheaves of ideals, will be called a *primary subscheme* of X_s. If Y is a primary subscheme of X_s, the union of all the irreducible primary subschemes not contained in Y will be called the *primary subscheme complementary to* Y and denoted as Y^c. By definition, the empty set and X_s are to be considered primary subschemes of X_s.

Let $\mathrm{Div}(X)$ denote the group of Cartier divisors of X, and $\mathrm{Div}^+(X)$ the submonoid of effective Cartier divisors. We will view an element of $\mathrm{Div}^+(X)$ as a closed subscheme of X. Conversely, we will write $Y \in \mathrm{Div}^+(X)$ for any closed subscheme of X defined locally everywhere by a nonzero-divisor.

For each closed subscheme Y of X, let $\mathcal{I}_Y$ denote its sheaf of ideals. If Z is another closed subscheme, we write $Y \leq Z$ if $Y \subseteq Z$. If $D \in \mathrm{Div}^+(X)$, let $Y + D$ denote the closed subscheme of X given by the sheaf of ideals $\mathcal{I}_D\mathcal{I}_Y$. In addition, if $D \subseteq Y$, let $Y - D$ denote the residual subscheme, given by the conductor ideal $(\mathcal{I}_Y : \mathcal{I}_D)$. Of course, $Y - D \leq Y \leq Y + D$ and $Y = (Y - D) + D = (Y + D) - D$.

Let $\mathrm{Twist}(f)$ denote the free Abelian group generated by $C_1, \dots, C_n$. An element of $\mathrm{Twist}(f)$ will be called a *twister*. We say that a twister $\gamma = \sum_i m_iC_i$ is *effective* if $m_i \geq 0$ for each $i = 1, \dots, n$, and *reduced* if, in addition, $m_i \leq 1$ for each $i = 1, \dots, n$. Let $\mathrm{Twist}^+(f) \subset \mathrm{Twist}(f)$ denote the submonoid of effective twisters. We can naturally identify the set of primary subschemes of X_s with the set of reduced effective twisters.

2.2. Modifications by primary subschemes

Let $\mathcal{J}$ be a coherent sheaf on X, and Y a primary subscheme of X_s. Let $\mathcal{J}_Y$ denote the restriction of $\mathcal{J}$ to Y modulo torsion. In other words, $\mathcal{J}_Y$ is the image of the natural map

$$\mathcal{J}|_Y \longrightarrow \bigoplus_{\xi_i \in Y} (\mathcal{J}|_Y)_{\xi_i}.$$

Let $\mathcal{J}(-Y)$ denote the kernel of the quotient map $\mathcal{J} \to \mathcal{J}_Y$. We say that $\mathcal{J}(-Y)$ is a *modification by Y of $\mathcal{J}$*. By definition, $\mathcal{J}_\emptyset = 0$ and $\mathcal{J}(-\emptyset) = \mathcal{J}$.

(We will never use the above construction for a sheaf denoted as $\mathcal{I}$. So, throughout the paper, $\mathcal{I}_Y$ will always be understood as the sheaf of ideals of a subscheme Y of X. If $\mathcal{J}$ is invertible along Y, and Y is a Cartier divisor, then $\mathcal{J}(-Y)$ is isomorphic to the twist of $\mathcal{J}$ by $-Y$. The standard notation for the twist will never be used in the paper, to avoid confusion.)

Notice that $\mathcal{J}(-Y)$ is also the kernel of the natural map

$$\mathcal{J} \longrightarrow \bigoplus_{\xi_i \in Y} (\mathcal{J}|_{X_s})_{\xi_i}.$$

So $\mathcal{J}(-Z) \subseteq \mathcal{J}(-Y)$ for each primary subscheme Z of X_s containing Y. In addition, $\pi\mathcal{J} \subseteq \mathcal{J}(-X_s)$. Hence, there is a natural map, $\mathcal{J} \to \mathcal{J}(-Y)$, obtained as the composition

$$\mathcal{J} \longrightarrow \pi\mathcal{J} \longrightarrow \mathcal{J}(-X_s) \longrightarrow \mathcal{J}(-Y),$$

where the first map is multiplication by π.

If $\mathcal{L}$ is an invertible sheaf on X, then $(\mathcal{J} \otimes \mathcal{L})(-Y) = \mathcal{J}(-Y) \otimes \mathcal{L}$, as subsheaves of $\mathcal{J} \otimes \mathcal{L}$.

The formation of the sheaves $\mathcal{J}_Y$ and $\mathcal{J}(-Y)$ is functorial on $\mathcal{J}$. Indeed, the composition of a map of coherent sheaves $\mathcal{J} \to \mathcal{K}$ with the natural quotient map $\mathcal{K} \to \mathcal{K}_Y$ factors through a map $\mathcal{J}_Y \to \mathcal{K}_Y$; thus the given map $\mathcal{J} \to \mathcal{K}$ takes $\mathcal{J}(-Y)$ to $\mathcal{K}(-Y)$. Also, the maps induced by a composition are the compositions of the corresponding induced maps.

Proposition 2.1. *Let $\mathcal{J}$ be a coherent sheaf on X. Then the following three statements hold.*

1. *For all primary subschemes Y and Z of X_s,*
$$\mathcal{J}(-Y)(-Z) = \mathcal{J}(-Z)(-Y).$$
2. *For all primary subschemes Y and Z of X_s such that $Z \subseteq Y^c$, the inclusions $\mathcal{J}(-Y) \to \mathcal{J}$ and $\mathcal{J}(-Z) \to \mathcal{J}$ induce injections $\mathcal{J}(-Y)_Z \to \mathcal{J}_Z$ and $\mathcal{J}(-Z)_Y \to \mathcal{J}_Y$ whose cokernels are isomorphic.*
3. *For all primary subschemes Y_1, Y_2 and Y_3 of X_s such that $Y_2 \subseteq Y_1^c$ and $Y_3 \subseteq Y_2^c$, the inclusion $\mathcal{J}(-Y_1 \cup Y_2) \to \mathcal{J}(-Y_1)$ induces a short exact sequence:*
$$0 \to \mathcal{J}(-Y_1 \cup Y_2)_{Y_3} \longrightarrow \mathcal{J}(-Y_1)_{Y_2 \cup Y_3} \longrightarrow \mathcal{J}(-Y_1)_{Y_2} \to 0.$$

Proof. Clearly, $\mathcal{I}(-Y)|_{X-Y} = \mathcal{I}|_{X-Y}$. In particular, the natural map

$$(\mathcal{I}(-Y)|_{X_s})_{\xi_i} \longrightarrow (\mathcal{I}|_{X_s})_{\xi_i}$$

is bijective for each $\xi_i \notin Y$. Therefore, $\mathcal{I}(-Y)(-Z) = \mathcal{I}(-Y \cup Z)$ if $Z \subseteq Y^c$. More generally, writing $Y = Y' \cup W$ and $Z = Z' \cup W$, where Y', Z' and W are primary subschemes such that $Y' \subseteq Z^c$ and $Z' \subseteq Y^c$, we have

$$\begin{aligned}\mathcal{I}(-Y)(-Z) &= \mathcal{I}(-Y')(-W)(-Z')(-W) = \mathcal{I}(-Y')(-Z)(-W)\\ &= \mathcal{I}(-Y' \cup Z)(-W) = \mathcal{I}(-Z)(-Y')(-W) = \mathcal{I}(-Z)(-Y).\end{aligned}$$

As for the second statement, since $\mathcal{I}(-Y)_{\xi_i} = \mathcal{I}_{\xi_i}$ for every $\xi_i \in Z$, it follows that the naturally induced map $\mathcal{I}(-Y)_Z \to \mathcal{I}_Z$ is injective. An analogous reasoning shows that $\mathcal{I}(-Z)_Y \to \mathcal{I}_Y$ is also injective. Now,

$$\frac{\mathcal{I}_Z}{\mathcal{I}(-Y)_Z} = \frac{\mathcal{I}/\mathcal{I}(-Z)}{\mathcal{I}(-Y)/\mathcal{I}(-Y)(-Z)} = \frac{\mathcal{I}}{\mathcal{I}(-Y) + \mathcal{I}(-Z)}.$$

By symmetry, $\mathcal{I}_Y/\mathcal{I}(-Z)_Y$ is thus isomorphic to $\mathcal{I}_Z/\mathcal{I}(-Y)_Z$.

As for the third statement, consider the natural short exact sequence:

$$0 \longrightarrow \frac{\mathcal{I}(-Y_1 \cup Y_2)}{\mathcal{I}(-Y_1 \cup Y_2)(-Y_3)} \longrightarrow \frac{\mathcal{I}(-Y_1)}{\mathcal{I}(-Y_1 \cup Y_2)(-Y_3)} \longrightarrow \frac{\mathcal{I}(-Y_1)}{\mathcal{I}(-Y_1 \cup Y_2)} \longrightarrow 0.$$

By definition, the first quotient is $\mathcal{I}(-Y_1 \cup Y_2)_{Y_3}$, while the last is $\mathcal{I}(-Y_1)_{Y_2}$. Now, using the first statement,

$$\mathcal{I}(-Y_1 \cup Y_2)(-Y_3) = \mathcal{I}(-Y_1)(-Y_2)(-Y_3) = \mathcal{I}(-Y_1)(-Y_2 \cup Y_3).$$

So, we may identify the middle quotient with $\mathcal{I}(-Y_1)_{Y_2 \cup Y_3}$, and thus obtain the desired short exact sequence. $\square$

2.3. *Modifications by twisters*

Let $\mathcal{I}$ be a coherent sheaf on X. For each $\gamma \in \mathrm{Twist}^+(f)$ define a subsheaf $\mathcal{I}^\gamma$ of $\mathcal{I}$ recursively as follows: if $\gamma = 0$, then $\mathcal{I}^\gamma := \mathcal{I}$; if $\gamma \neq 0$, then let

$$\mathcal{I}^\gamma := \mathcal{I}^{\gamma - C_i}(-C_i)$$

for any C_i such that $\gamma - C_i$ is effective. It follows from the first statement of Proposition 2.1 that $\mathcal{I}^\gamma$ is well-defined, and

$$\mathcal{I}^{\gamma_1+\gamma_2} = (\mathcal{I}^{\gamma_1})^{\gamma_2}$$

for every two $\gamma_1, \gamma_2 \in \mathrm{Twist}^+(f)$. We call $\mathcal{I}^\gamma$ the *γ-modification of $\mathcal{I}$*. Sometimes we will also use the notation $\mathcal{I}(-\gamma)$ instead of $\mathcal{I}^\gamma$.

If $\mathcal{L}$ is an invertible sheaf on X, then $\mathcal{I}^\gamma \otimes \mathcal{L} = (\mathcal{I} \otimes \mathcal{L})^\gamma$ as subsheaves of $\mathcal{I} \otimes \mathcal{L}$. Also, the γ-modifications $\mathcal{I}^\gamma$ are functorial on $\mathcal{I}$, in the sense explained in Section 2.2.

2.4. *Torsion-free, rank-1 sheaves.*

Let $\mathcal{I}$ be an S-flat coherent sheaf on X. We say that $\mathcal{I}$ is *torsion-free on X/S* if the associated components of $\mathcal{I}|_{X_s}$ are components of X_s, or equivalently, if the natural map $\mathcal{I}|_{X_s} \to \mathcal{I}_{X_s}$ is a bijection. We say that $\mathcal{I}$ is *of rank 1 on X/S* if $(\mathcal{I}|_{X_s})_{\xi_i} \cong \mathcal{O}_{X_s,\xi_i}$ for each $i = 1, \ldots, n$.

Proposition 2.2. *Let $\mathcal{I}$ be a torsion-free sheaf on X/S, and Y a primary subscheme of X_s. Set $Z := Y^c$. Then the following four statements hold.*

1. *$\pi\mathcal{I} = \mathcal{I}(-X_s)$ and the natural maps*

   $$\mathcal{I}|_{X_s} \longrightarrow \mathcal{I}_{X_s} \quad \text{and} \quad \mathcal{I} \longrightarrow \pi\mathcal{I}$$

   are isomorphisms.
2. *$\mathcal{I}(-Y)$ is torsion-free on X/S.*
3. *The natural maps $\mathcal{I}(-Y) \to \mathcal{I}$ and $\mathcal{I} \to \mathcal{I}(-Y)$ are injective and induce short exact sequences:*

   $$\begin{aligned}&0 \to \mathcal{I}(-Y)_Z \to \mathcal{I}|_{X_s} \to \mathcal{I}_Y \to 0,\\ &0 \to \mathcal{I}_Y \to \mathcal{I}(-Y)|_{X_s} \to \mathcal{I}(-Y)_Z \to 0.\end{aligned}$$

4. *If $\mathcal{I}$ is of rank 1, so is $\mathcal{I}(-Y)$.*

Proof. The first map, $\mathcal{J}|_{X_s} \to \mathcal{J}_{X_s}$, is an isomorphism because the associated components of $\mathcal{J}|_{X_s}$ are components of X_s. Thus $\mathcal{J}(-X_s) = \pi\mathcal{J}$. In addition, since $\mathcal{J}$ is S-flat, the multiplication-by-π map $\mathcal{J} \to \pi\mathcal{J}$ is an isomorphism.

As for the second statement, since S is the spectrum of a discrete valuation ring, and $\mathcal{J}$ is S-flat, its subsheaf $\mathcal{J}(-Y)$ is also S-flat. In addition, since the multiplication-by-π bijection $\mathcal{J} \to \pi\mathcal{J}$ carries $\mathcal{J}(-Y)$ onto $(\pi\mathcal{J})(-Y)$, by functoriality, we have

$$\mathcal{J}(-Y)(-X_s) = \mathcal{J}(-X_s)(-Y) = (\pi\mathcal{J})(-Y) = \pi\mathcal{J}(-Y),$$

and thus the natural map $\mathcal{J}(-Y)|_{X_s} \to \mathcal{J}(-Y)_{X_s}$ is an isomorphism. So $\mathcal{J}(-Y)$ is torsion-free.

Consider now the third statement. Recall that the natural map $\mathcal{J}(-Y) \to \mathcal{J}$ is simply an inclusion, and hence injective. Also, the natural map $\mathcal{J} \to \mathcal{J}(-Y)$ is injective if and only if the multiplication-by-π map $\mathcal{J} \to \pi\mathcal{J}$ is an isomorphism, and this is the case by the first statement.

As for the exact sequences, the first is obtained from that in Proposition 2.1 by setting $Y_1 := \emptyset$, $Y_2 := Y$ and $Y_3 := Z$, and recalling from the first statement that $\mathcal{J}|_{X_s} = \mathcal{J}_{X_s}$.

The second is also obtained from that in Proposition 2.1, this time by setting $Y_1 := Y$, $Y_2 := Z$ and $Y_3 := Y$. However, we use the composition of isomorphisms,

$$\mathcal{J} \longrightarrow \pi\mathcal{J} \longrightarrow \mathcal{J}(-X_s),$$

to replace the leftmost sheaf $\mathcal{J}(-X_s)_Y$ with $\mathcal{J}_Y$, and we use that $\mathcal{J}(-Y)$ is torsion-free, to replace $\mathcal{J}(-Y)_{X_s}$ with $\mathcal{J}(-Y)|_{X_s}$.

The fourth statement follows from the two exact sequences of the third statement. Indeed, the first one yields $(\mathcal{J}(-Y)|_{X_s})_{\xi_i} \cong (\mathcal{J}|_{X_s})_{\xi_i}$ for each $\xi_i \in Z$, while the second one yields $(\mathcal{J}|_{X_s})_{\xi_i} \cong (\mathcal{J}(-Y)|_{X_s})_{\xi_i}$ for each $\xi_i \in Y$. Thus $\mathcal{J}(-Y)$ is of rank 1 if and only if so is $\mathcal{J}$. $\square$

3. Limits of Cartier divisors

3.1. Cycles

Assume X_s is of pure dimension, say d. Let $\mathrm{Cyc}(X_s)$ denote the free Abelian group generated by all integral closed subschemes of X_s of dimension $d-1$. We will simply say that an element of $\mathrm{Cyc}(X_s)$ is a *cycle*. A cycle is called *effective* if its expression as a $\mathbf{Z}$-linear combination of integral subschemes involves only nonnegative coefficients. Let $\mathrm{Cyc}^+(X_s) \subset \mathrm{Cyc}(X_s)$ denote the submonoid of effective cycles.

For any coherent sheaf $\mathcal{F}$ on X_s with support of dimension at most $d-1$, let

$$[\mathcal{F}] := \sum_Y \ell(\mathcal{F}_{\xi_Y})[Y] \in \mathrm{Cyc}^+(X_s),$$

where the sum runs over all irreducible components Y of dimension $d-1$ of the support of $\mathcal{F}$, with ξ_Y denoting the generic point of Y. Since localization is exact and length is additive, if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of coherent sheaves on X_s with support of dimension at most $d-1$, then $[\mathcal{G}] = [\mathcal{F}] + [\mathcal{H}]$; so we say that "the bracket is additive". Notice as well that, if $\mathcal{L}$ is an invertible sheaf on X_s, then $[\mathcal{F} \otimes \mathcal{L}] = [\mathcal{F}]$.

If $W \subset X_s$ is a closed subscheme of dimension at most $d-1$, let $[W] := [\mathcal{O}_W]$. We call $[W]$ the *fundamental cycle* of W.

Lemma 3.1. *Assume X_s is of pure dimension. Let $\mathcal{F}$ be a coherent sheaf on X and $\mathcal{G} \subseteq \mathcal{F}$ a coherent subsheaf. Let $C_{i_1}, \ldots, C_{i_m}$ be a collection of distinct irreducible primary subschemes of X_s. Suppose $\mathcal{F}_{\xi_{i_j}} = \mathcal{G}_{\xi_{i_j}}$ for each $j = 1, \ldots, m$. Set*

$$Z_j := \bigcup_{\ell=1}^{j} C_{i_\ell} \quad \text{and} \quad Z'_j := \bigcup_{\ell=j+1}^{m} C_{i_\ell}$$

for each $j = 0, \ldots, m$. Then

$$\left[\frac{\mathcal{F}_{Z_m}}{\mathcal{G}_{Z_m}}\right] = \sum_{j=0}^{m-1} \left[\frac{\mathcal{F}(-Z_j)_{C_{i_{j+1}}}}{\mathcal{G}(-Z_j)_{C_{i_{j+1}}}}\right]. \tag{3.1.1}$$

Proof. For each $j = 0, \ldots, m-1$, apply the third statement of Proposition 2.1 with $Y_1 := Z_j$, $Y_2 := C_{i_{j+1}}$ and $Y_3 := Z'_{j+1}$ to both $\mathcal{J} := \mathcal{F}$ and $\mathcal{J} := \mathcal{G}$. By functoriality, we get a natural map of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}(-Z_{j+1})_{Z'_{j+1}} & \longrightarrow & \mathcal{G}(-Z_j)_{Z'_j} & \longrightarrow & \mathcal{G}(-Z_j)_{C_{i_{j+1}}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}(-Z_{j+1})_{Z'_{j+1}} & \longrightarrow & \mathcal{F}(-Z_j)_{Z'_j} & \longrightarrow & \mathcal{F}(-Z_j)_{C_{i_{j+1}}} & \longrightarrow & 0.
\end{array}$$

The vertical map to the right is always injective with cokernel supported in codimension 1 in $C_{i_{j+1}}$, because $\mathcal{G}_{\xi_{i_{j+1}}} = \mathcal{F}_{\xi_{i_{j+1}}}$. Thus, all the vertical maps are injective with cokernel supported in codimension 1 in X_s, and (3.1.1) holds by the snake lemma. $\square$

3.2. Limits of Cartier divisors

Assume that $f : X \to S$ is flat, or equivalently, that X_s is a Cartier divisor of X. Assume as well that X_s is of pure dimension. Since X_s is a Cartier divisor of X, also X, and thus X_η, is of pure dimension. For each closed subscheme D of X, let

$$\lim D := X_s \cap \overline{D \cap X_\eta}^X.$$

We call $\lim D$ the *limit subscheme of D*.

Suppose $D \cap X_\eta$ is a Cartier divisor. Since X_η is of pure dimension, $D \cap X_\eta$ is of pure codimension 1 in X_η. Thus, since $\overline{D \cap X_\eta}^X$ is S-flat, also $\lim D$ is of pure codimension 1 in X_s. Let $[\lim D]$ denote the associated cycle. We call $[\lim D]$ the *limit cycle of D*.

Proposition 3.2. *Assume that $f : X \to S$ is flat and X_s has pure dimension. Let D_1, D_2 and D_3 be S-flat closed subschemes of X of pure codimension* 1. *Assume that $D_1 \cap X_\eta$ is a Cartier divisor of X_η and*

$$D_3 \cap X_\eta = (D_1 \cap X_\eta) + (D_2 \cap X_\eta).$$

Then

$$[D_3 \cap X_s] = [D_1 \cap X_s] + [D_2 \cap X_s].$$

(This proposition is a slight generalization of [6], Prop. 5.12, p. 49.)

Proof. For each $i = 1, 2, 3$, since D_i is S-flat of pure codimension 1, also $D_i \cap X_s$ is of pure codimension 1 in X_s. Again by flatness, D_i is the closure of $D_i \cap X_\eta$. Thus, from the hypotheses, we get that $D_3 = D_1 \cup D_2$ set-theoretically, and hence

$$D_3 \cap X_s = (D_1 \cap X_s) \cup (D_2 \cap X_s)$$

set-theoretically.

Let $W \subseteq D_3 \cap X_s$ be an irreducible component. We need only show that the coefficient of $[W]$ in the expression for $[D_3 \cap X_s]$ is the sum of those for $[D_1 \cap X_s]$ and $[D_2 \cap X_s]$. Let $\zeta \in W$ be the generic point, and set $A := \mathcal{O}_{X,\zeta}$. Let I_1, I_2 and I_3 be the respective ideals of D_1, D_2 and D_3 in A.

Let $Y_1, \dots, Y_r$ be the irreducible components of D_3 containing W. These correspond to the minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ of A containing I_3. Notice that, for $i = 1, 2$, since D_i and D_3 have the same pure dimension, and $D_3 \supseteq D_i$, the minimal prime ideals of A containing I_i are those $\mathfrak{p}_j$ such that $\mathfrak{p}_j \supseteq I_i$.

Since D_i is S-flat, π is a nonzero-divisor of A/I_i for each $i = 1, 2, 3$. In particular, $\pi \notin \mathfrak{p}_j$ for any j. By [5], Lemme 21.10.17.7, p. 299 or [4], Lemma A.2.7, p. 410, for $i = 1, 2, 3$,

$$\ell(A/(I_i + \pi A)) = \sum_{j=1}^{r} \ell(A_{\mathfrak{p}_j}/I_i A_{\mathfrak{p}_j})\ell(A/(\mathfrak{p}_j + \pi A)). \tag{3.2.1}$$

The left-hand side of (3.2.1) is the coefficient of $[W]$ in the expression for the cycle $[D_i \cap X_s]$. Thus, we need only show that, for each $j = 1, \dots, r$,

$$\ell(A_{\mathfrak{p}_j}/I_3 A_{\mathfrak{p}_j}) = \ell(A_{\mathfrak{p}_j}/I_1 A_{\mathfrak{p}_j}) + \ell(A_{\mathfrak{p}_j}/I_2 A_{\mathfrak{p}_j}). \tag{3.2.2}$$

Now, since $\pi \notin \mathfrak{p}_j$, we have $I_i A_{\mathfrak{p}_j} = I_i A_\pi A_{\mathfrak{p}_j}$ for $i = 1, 2, 3$. By the hypotheses of the proposition, $I_3 A_\pi = I_1 I_2 A_\pi$, and there is a nonzero-divisor $f_j \in A_{\mathfrak{p}_j}$ such that $I_1 A_{\mathfrak{p}_j} = f_j A_{\mathfrak{p}_j}$. Since f_j is not a zero-divisor, multiplication by f_j induces a short exact sequence:

$$0 \to \frac{A_{\mathfrak{p}_j}}{I_2 A_{\mathfrak{p}_j}} \longrightarrow \frac{A_{\mathfrak{p}_j}}{f_j I_2 A_{\mathfrak{p}_j}} \longrightarrow \frac{A_{\mathfrak{p}_j}}{f_j A_{\mathfrak{p}_j}} \to 0.$$

Since the length is additive, we get (3.2.2). $\square$

4. The main theorem

Theorem 4.1. *Assume that $f : X \to S$ is flat and X_s has pure dimension. Let D be an effective Cartier divisor of X. Suppose that, for each $i = 1, \dots, n$, there are effective Cartier divisors E_i and F_i of X and a nonnegative integer p_i such that $\xi_i \notin E_i + F_i$ and $D + E_i = p_i X_s + F_i$. Then*

$$[\lim D] = \sum_{i=1}^{n} \Big([F_i \cap C_i] - [E_i \cap C_i]\Big).$$

(Recall that the ξ_i are the generic points of the C_i, the irreducible primary subschemes of the special fiber X_s.)

Proof. We will define in (4.1.1) an ideal sheaf $\mathcal{J}$ which will be used to translate the hypothesis on the Cartier divisors E_i and F_i into the formulas (4.1.6) and (4.1.9). It will turn out from (4.1.6) that $\mathcal{J}$ is in fact the ideal sheaf of $\overline{D \cap X_\eta}^X$. Then we shall use Lemma 3.1 to give a formula, (4.1.11), for $[\lim D]$ involving $\mathcal{J}$, which will be simplified by means of (4.1.9) in such a way that we are reduced to proving a claim that does not involve D any longer. The claim is reduced by induction to (4.1.13), which is then proved by applying Lemma 3.1 and the second statement of Proposition 2.1.

To start with, let $\gamma = \sum_i r_i C_i$ be an effective twister for which $\mathcal{I}_D \subseteq \mathcal{O}_X^\gamma$. Suppose $r := r_1 + \cdots + r_n$ is maximal for this property. (There exists such γ because $\ell(\mathcal{O}_{D,\xi_i}) < \infty$ for each $i = 1, \ldots, n$.) Then, since X is S-flat,

$$\mathcal{I}_D\Big(-\sum_i \sum_{j \neq i} r_i C_j\Big) \subseteq \mathcal{O}_X(-r(C_1 + \cdots + C_n)) = \pi^r \mathcal{O}_X.$$

Let

$$\mathcal{J} := \left(\mathcal{I}_D\Big(-\sum_{i=1}^n \sum_{j \neq i} r_i C_j\Big) : \pi^r \mathcal{O}_X\right) \subseteq \mathcal{O}_X. \tag{4.1.1}$$

Since f is flat, π is a nonzero-divisor of $\mathcal{O}_X$, and thus

$$\pi^r \mathcal{J} = \mathcal{I}_D\Big(-\sum_{i=1}^n \sum_{j \neq i} r_i C_j\Big). \tag{4.1.2}$$

It follows that

$$\pi^r \mathcal{J}^\gamma = (\pi^r \mathcal{J})(-\gamma) = \mathcal{I}_D(-r(C_1 + \cdots + C_n)) = \pi^r \mathcal{I}_D,$$

whence $\mathcal{J}^\gamma = \mathcal{I}_D$. Note that, since $\mathcal{I}_D$ is invertible, and hence torsion-free of rank 1 on X/S, it follows from (4.1.2) and Proposition 2.2 that also $\pi^r \mathcal{J}$ is torsion-free of rank 1, and hence so is the isomorphic sheaf $\mathcal{J}$. In addition, $\mathcal{J} \not\subseteq \mathcal{O}_X(-C_i)$ for every $i = 1, \ldots, n$, by the maximality of r. After reordering the components C_i, we may assume that

$$r_1 \geq r_2 \geq \cdots \geq r_{n-1} \geq r_n. \tag{4.1.3}$$

If $\mathcal{K}$ is a sheaf of ideals and G is a Cartier divisor of X, then the multiplication map $\mathcal{K} \otimes \mathcal{I}_G \to \mathcal{K}\mathcal{I}_G$ is an isomorphism. Thus, for each $\mu \in \mathrm{Twist}^+(f)$, since $(\mathcal{K} \otimes \mathcal{I}_G)^\mu = \mathcal{K}^\mu \otimes \mathcal{I}_G$ as subsheaves of $\mathcal{K} \otimes \mathcal{I}_G$, and since the multiplication map carries $\mathcal{K}^\mu \otimes \mathcal{I}_G$ onto $\mathcal{K}^\mu \mathcal{I}_G$ and $(\mathcal{K} \otimes \mathcal{I}_G)^\mu$ onto $(\mathcal{K}\mathcal{I}_G)^\mu$, we have

$$\mathcal{K}^\mu \mathcal{I}_G = (\mathcal{K}\mathcal{I}_G)^\mu. \tag{4.1.4}$$

In addition, if $G \cap Y$ is Cartier for a primary subscheme Y, then $\mathcal{I}_G|_Y = \mathcal{I}_{G \cap Y|Y}$, and it follows that

$$(\mathcal{K}\mathcal{I}_G)_Y = \mathcal{K}_Y \mathcal{I}_{G \cap Y|Y}. \tag{4.1.5}$$

By the hypothesis of the theorem, for each $i = 1, \ldots, n$,

$$\pi^{p_i} \mathcal{I}_{F_i} = \mathcal{I}_{p_i X_s + F_i} = \mathcal{I}_{D + E_i} = \mathcal{I}_D \mathcal{I}_{E_i} = \mathcal{J}^\gamma \mathcal{I}_{E_i}.$$

From Eq. (4.1.4), it follows that $\pi^{p_i} \mathcal{I}_{F_i} = (\mathcal{J}\mathcal{I}_{E_i})^\gamma$. Now, since $\xi_i \notin E_i$ and $\mathcal{J} \not\subseteq \mathcal{O}_X(-C_i)$, the largest integer j such that $(\mathcal{J}\mathcal{I}_{E_i})^\gamma \subseteq \mathcal{O}_X(-jC_i)$ is r_i. On the other hand, since also $\xi_i \notin F_i$, the largest integer j such that $\pi^{p_i} \mathcal{I}_{F_i} \subseteq \mathcal{O}_X(-jC_i)$ is p_i. Since $\pi^{p_i} \mathcal{I}_{F_i} = (\mathcal{J}\mathcal{I}_{E_i})^\gamma$, we have $p_i = r_i$. Putting

$$\alpha_i := \sum_{j<i} (r_j - r_i) C_j \quad \text{and} \quad \beta_i := \sum_{j>i} (r_i - r_j) C_j,$$

we get

$$(\mathcal{J}\mathcal{I}_{E_i})^{\alpha_i} = \mathcal{I}_{F_i}^{\beta_i}. \tag{4.1.6}$$

Let

$$\gamma' := \sum_j r'_j C_j, \quad \text{where } r'_j := r_1 - r_j \text{ for each } j = 1, \ldots, n, \tag{4.1.7}$$

and set

$$\delta_i := \sum_{j<i} r'_j C_j + \sum_{j \geq i} r'_i C_j, \quad \text{and} \quad \epsilon_i := \sum_{j<i} r'_i C_j + \sum_{j \geq i} r'_j C_j. \tag{4.1.8}$$

Then $\alpha_i + \delta_i = r'_i(C_1 + \cdots + C_n)$ and $\beta_i + \delta_i = \gamma'$. Since $\mathcal{J}\mathcal{I}_{E_i}$ is torsion-free of rank 1, it follows from (4.1.6) that

$$\pi^{r'_i} \mathcal{J}\mathcal{I}_{E_i} = (\mathcal{J}\mathcal{I}_{E_i})^{\alpha_i + \delta_i} = ((\mathcal{J}\mathcal{I}_{E_i})^{\alpha_i})^{\delta_i} = (\mathcal{I}_{F_i}^{\beta_i})^{\delta_i} = \mathcal{I}_{F_i}^{\beta_i + \delta_i} = \mathcal{I}_{F_i}^{\gamma'}. \tag{4.1.9}$$

Notice, for later use, that $\epsilon_i = \alpha_i + \gamma'$.

Since $\xi_i \notin E_i + F_i$, and since C_i is not a summand of α_i or β_i, it follows from (4.1.6) that

$$\mathcal{J}_{\xi_i} = (\mathcal{J}\mathcal{I}_{E_i})_{\xi_i} = ((\mathcal{J}\mathcal{I}_{E_i})^{\alpha_i})_{\xi_i} = (\mathcal{I}_{F_i}^{\beta_i})_{\xi_i} = (\mathcal{I}_{F_i})_{\xi_i} = \mathcal{O}_{X,\xi_i}.$$

Since this holds for each $i = 1, \dots, n$, and since $\mathcal{J}$ is torsion-free on X/S, it follows that the induced map $\mathcal{J}|_{X_s} \to \mathcal{O}_{X_s}$ is injective. So the inclusion $\mathcal{J} \to \mathcal{O}_X$ has flat cokernel over S. Since $\mathcal{J}^\gamma = \mathcal{I}_D$, we have that $\mathcal{J}|_{X_\eta}$ is the sheaf of ideals of $D \cap X_\eta$ in X_η. Thus $\lim D$ is the subscheme of X_s with ideal sheaf $\mathcal{J}|_{X_s}$, and hence

$$[\lim D] = \left[\frac{\mathcal{O}_{X_s}}{\mathcal{J}|_{X_s}}\right]. \tag{4.1.10}$$

Set $Z_1 := \emptyset$ and, for each $j = 2, \dots, n+1$, put $Z_j := C_1 \cup \dots \cup C_{j-1}$. By Lemma 3.1,

$$\left[\frac{\mathcal{O}_{X_s}}{\mathcal{J}|_{X_s}}\right] = \sum_{i=1}^n \left[\frac{\mathcal{O}_X(-Z_i)_{C_i}}{\mathcal{J}(-Z_i)_{C_i}}\right],$$

and thus, by (4.1.10), the bracket being additive,

$$[\lim D] = \sum_{i=1}^n \left(\left[\frac{\mathcal{O}_{C_i}}{\mathcal{J}_{C_i}}\right] - \left[\frac{\mathcal{O}_{C_i}}{\mathcal{O}_X(-Z_i)_{C_i}}\right] + \left[\frac{\mathcal{J}_{C_i}}{\mathcal{J}(-Z_i)_{C_i}}\right]\right). \tag{4.1.11}$$

Now, using (4.1.4), (4.1.5) and (4.1.9), and using that both $C_i \cap E_i$ and $C_i \cap F_i$ are Cartier, we get

$$\begin{aligned}
\left[\frac{\mathcal{J}_{C_i}}{\mathcal{J}(-Z_i)_{C_i}}\right] &= \left[\frac{\mathcal{J}_{C_i}\mathcal{I}_{E_i\cap C_i|C_i}}{\mathcal{J}(-Z_i)_{C_i}\mathcal{I}_{E_i\cap C_i|C_i}}\right] = \left[\frac{(\mathcal{J}\mathcal{I}_{E_i})_{C_i}}{(\mathcal{J}\mathcal{I}_{E_i})(-Z_i)_{C_i}}\right] \\
&= \left[\frac{(\pi^{r_i'}\mathcal{J}\mathcal{I}_{E_i})_{C_i}}{(\pi^{r_i'}\mathcal{J}\mathcal{I}_{E_i})(-Z_i)_{C_i}}\right] = \left[\frac{(\mathcal{I}_{F_i}^{\gamma'})_{C_i}}{\mathcal{I}_{F_i}^{\gamma'}(-Z_i)_{C_i}}\right] \\
&= \left[\frac{(\mathcal{O}_X^{\gamma'}\mathcal{I}_{F_i})_{C_i}}{(\mathcal{O}_X^{\gamma'}\mathcal{I}_{F_i})(-Z_i)_{C_i}}\right] = \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_i}\mathcal{I}_{F_i\cap C_i|C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}\mathcal{I}_{F_i\cap C_i|C_i}}\right] \\
&= \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right].
\end{aligned}$$

Thus (4.1.11) becomes

$$[\lim D] = \sum_{i=1}^n \left(\left[\frac{\mathcal{O}_{C_i}}{\mathcal{J}_{C_i}}\right] - \left[\frac{\mathcal{O}_{C_i}}{\mathcal{O}_X(-Z_i)_{C_i}}\right] + \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right]\right). \tag{4.1.12}$$

Using a similar reasoning, since the bracket is additive,

$$\begin{aligned}
\left[\frac{\mathcal{O}_{C_i}}{\mathcal{J}_{C_i}}\right] &= \left[\frac{(\mathcal{I}_{E_i})_{C_i}}{(\mathcal{J}\mathcal{I}_{E_i})_{C_i}}\right] = \left[\frac{\mathcal{O}_{C_i}}{(\mathcal{J}\mathcal{I}_{E_i})_{C_i}}\right] - [E_i \cap C_i] \\
&= \left[\frac{(\pi^{r_i'}\mathcal{O}_X)_{C_i}}{(\pi^{r_i'}\mathcal{J}\mathcal{I}_{E_i})_{C_i}}\right] - [E_i \cap C_i] = \left[\frac{(\pi^{r_i'}\mathcal{O}_X)_{C_i}}{(\mathcal{I}_{F_i}^{\gamma'})_{C_i}}\right] - [E_i \cap C_i] \\
&= [F_i \cap C_i] + \left[\frac{(\pi^{r_i'}\mathcal{I}_{F_i})_{C_i}}{(\mathcal{I}_{F_i}^{\alpha_i+\gamma'})_{C_i}}\right] - \left[\frac{(\mathcal{I}_{F_i}^{\gamma'})_{C_i}}{(\mathcal{I}_{F_i}^{\alpha_i+\gamma'})_{C_i}}\right] - [E_i \cap C_i] \\
&= [F_i \cap C_i] - [E_i \cap C_i] + \left[\frac{(\pi^{r_i'}\mathcal{O}_X)_{C_i}}{(\mathcal{O}_X^{\epsilon_i})_{C_i}}\right] - \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_i}}{(\mathcal{O}_X^{\epsilon_i})_{C_i}}\right],
\end{aligned}$$

where we used that $\epsilon_i = \alpha_i + \gamma'$. Substituting in (4.1.12), we see that we need only prove the following claim.

Claim: Let $\gamma := \sum_i r_i C_i$ be an effective twister such that (4.1.3) holds. Let γ' be as in (4.1.7). For each $i = 1, \dots, n$, let $\epsilon_i \in \mathrm{Twist}(f)$ be as in (4.1.8), and put

$$\theta_i(\gamma) := \left[\frac{(\pi^{r_i'}\mathcal{O}_X)_{C_i}}{(\mathcal{O}_X^{\epsilon_i})_{C_i}}\right] - \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_i}}{(\mathcal{O}_X^{\epsilon_i})_{C_i}}\right] - \left[\frac{\mathcal{O}_{C_i}}{\mathcal{O}_X(-Z_i)_{C_i}}\right] + \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right].$$

Then $\theta_1(\gamma) + \dots + \theta_n(\gamma) = 0$.

We will prove the claim by induction on the sum $r' := r'_1 + \cdots + r'_n$. If $r' = 0$, then $\gamma' = 0$ and $\epsilon_i = 0$ for each $i = 1, \ldots, n$. The claim is trivial in this case, as the first and second summands of $\theta_i(\gamma)$ are zero, and the third and fourth cancel each other, for each $i = 1, \ldots, n$.

Now, suppose $r' > 0$. Then one of the inequalities in (4.1.3) is strict. Let ℓ be an integer, between 2 and n, such that $r_{\ell-1} > r_\ell$. For each $i = 1, \ldots, n$, let $t_i := r_i$ if $i \neq \ell$ and $t_\ell := r_\ell + 1$. Then $t_1 \geq \cdots \geq t_n$ as well. Set $\tau := \sum_i t_i C_i$ and $\tau' := \sum_i t'_i C_i$, where $t'_i := t_1 - t_i$ for each $i = 1, \ldots, n$. Then $t'_i = r'_i$ for every $i \neq \ell$, but $t'_\ell = r'_\ell - 1$. So, by induction, $\theta_1(\tau) + \cdots + \theta_n(\tau) = 0$.

Set

$$\rho_i := \sum_{j<i} t'_i C_j + \sum_{j \geq i} t'_j C_j \quad \text{for } i = 1, \ldots, n.$$

Then $\gamma' = \tau' + C_\ell$ and

$$\epsilon_i = \begin{cases} \rho_i + C_\ell & \text{if } i < \ell, \\ \rho_\ell + C_1 + \cdots + C_\ell & \text{if } i = \ell, \\ \rho_i & \text{if } i > \ell. \end{cases}$$

Using the above formulas, and the fact that the bracket is additive, we get

$$\theta_i(\gamma) - \theta_i(\tau) = \left[\frac{\mathcal{O}_X^{\tau'}(-Z_i)_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right] \quad \text{if } i \neq \ell,$$

and

$$\begin{aligned} \theta_\ell(\gamma) - \theta_\ell(\tau) &= \left[\frac{(\pi^{r'_\ell}\mathcal{O}_X)_{C_\ell}}{(\mathcal{O}_X^{\epsilon_\ell})_{C_\ell}}\right] - \left[\frac{(\pi^{t'_\ell}\mathcal{O}_X)_{C_\ell}}{(\mathcal{O}_X^{\rho_\ell})_{C_\ell}}\right] - \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_\ell}}{(\mathcal{O}_X^{\epsilon_\ell})_{C_\ell}}\right] + \left[\frac{(\mathcal{O}_X^{\tau'})_{C_\ell}}{(\mathcal{O}_X^{\rho_\ell})_{C_\ell}}\right] + \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_\ell}}{\mathcal{O}_X^{\gamma'}(-Z_\ell)_{C_\ell}}\right] - \left[\frac{(\mathcal{O}_X^{\tau'})_{C_\ell}}{\mathcal{O}_X^{\tau'}(-Z_\ell)_{C_\ell}}\right] \\ &= \left[\frac{(\pi^{r'_\ell}\mathcal{O}_X)_{C_\ell}}{(\mathcal{O}_X^{\epsilon_\ell})_{C_\ell}}\right] - \left[\frac{(\pi^{r'_\ell}\mathcal{O}_X)_{C_\ell}}{(\pi\mathcal{O}_X^{\rho_\ell})_{C_\ell}}\right] - \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_\ell}}{(\mathcal{O}_X^{\epsilon_\ell})_{C_\ell}}\right] + \left[\frac{(\pi\mathcal{O}_X^{\tau'})_{C_\ell}}{(\pi\mathcal{O}_X^{\rho_\ell})_{C_\ell}}\right] + \left[\frac{(\mathcal{O}_X^{\gamma'})_{C_\ell}}{\mathcal{O}_X^{\gamma'}(-Z_\ell)_{C_\ell}}\right] - \left[\frac{(\mathcal{O}_X^{\tau'})_{C_\ell}}{\mathcal{O}_X^{\tau'}(-Z_\ell)_{C_\ell}}\right] \\ &= -\left[\frac{\mathcal{O}_X^{\gamma'}(-Z_\ell)_{C_\ell}}{(\pi\mathcal{O}_X^{\tau'})_{C_\ell}}\right] - \left[\frac{(\mathcal{O}_X^{\tau'})_{C_\ell}}{\mathcal{O}_X^{\tau'}(-Z_\ell)_{C_\ell}}\right]. \end{aligned}$$

Thus, since $\theta_1(\tau) + \cdots + \theta_n(\tau) = 0$ by induction, we need only show that

$$\sum_{i \neq \ell}\left[\frac{\mathcal{O}_X^{\tau'}(-Z_i)_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right] = \left[\frac{\mathcal{O}_X^{\gamma'}(-Z_\ell)_{C_\ell}}{(\pi\mathcal{O}_X^{\tau'})_{C_\ell}}\right] + \left[\frac{(\mathcal{O}_X^{\tau'})_{C_\ell}}{\mathcal{O}_X^{\tau'}(-Z_\ell)_{C_\ell}}\right]. \tag{4.1.13}$$

Now, applying Lemma 3.1 twice, we get that

$$\begin{aligned} \sum_{i \neq \ell}\left[\frac{\mathcal{O}_X^{\tau'}(-Z_i)_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right] &= \sum_{i<\ell}\left[\frac{\mathcal{O}_X^{\tau'}(-Z_i)_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right] + \sum_{i>\ell}\left[\frac{\mathcal{O}_X^{\tau'}(-Z_i)_{C_i}}{\mathcal{O}_X^{\gamma'}(-Z_i)_{C_i}}\right] \\ &= \left[\frac{(\mathcal{O}_X^{\tau'})_{Z_\ell}}{(\mathcal{O}_X^{\gamma'})_{Z_\ell}}\right] + \left[\frac{\mathcal{O}_X^{\tau'}(-Z_{\ell+1})_{Z^c_{\ell+1}}}{\mathcal{O}_X^{\gamma'}(-Z_{\ell+1})_{Z^c_{\ell+1}}}\right]. \end{aligned}$$

In addition, since $\gamma' = \tau' + C_\ell$, it follows from the second statement of Proposition 2.1 that

$$\left[\frac{(\mathcal{O}_X^{\tau'})_{Z_\ell}}{(\mathcal{O}_X^{\gamma'})_{Z_\ell}}\right] = \left[\frac{(\mathcal{O}_X^{\tau'})_{C_\ell}}{\mathcal{O}_X^{\tau'}(-Z_\ell)_{C_\ell}}\right],$$

$$\left[\frac{\mathcal{O}_X^{\tau'}(-Z_{\ell+1})_{Z^c_{\ell+1}}}{\mathcal{O}_X^{\gamma'}(-Z_{\ell+1})_{Z^c_{\ell+1}}}\right] = \left[\frac{\mathcal{O}_X^{\tau'}(-Z_{\ell+1})_{C_\ell}}{(\pi\mathcal{O}_X^{\tau'})_{C_\ell}}\right].$$

Eq. (4.1.13) follows. □

Remark 4.2. Our setup is a little more general, but it seems that a variation of [4], Prop. 11.1, p. 196, can be used to show that the formula of Theorem 4.1 above holds in the Chow group of cycle classes of X_s. Even so, the theorem is stronger as it gives a formula for the cycle itself.

Proposition 4.3. *Assume that $f: X \to S$ is projective and flat, and that S is the spectrum of a discrete valuation ring containing an infinite field k. Then the following two statements hold:*

1. *For each $i = 1, \dots, n$ there is $G_i \in \mathrm{Div}^+(X)$ such that $\xi_j \notin G_i$ for $j \neq i$ and G_i coincides with C_i at ξ_i.*
2. *If X_s is reduced, for each closed subscheme D of X such that $D \cap X_\eta$ is a Cartier divisor there are divisors $E_i \in \mathrm{Div}^+(X)$ with $\xi_i \notin E_i$ and nonnegative integers p_i such that $D + E_i \geq p_i X_s$ and $\xi_i \notin (D + E_i) - p_i X_s$ for $i = 1, \dots, n$.*

Proof. Let R be the ring of regular functions of S. Since S is a k-scheme, R is a k-algebra. Since f is projective, f factors through an embedding $\iota\colon X \to \mathbf{P}^m_R$, where $\mathbf{P}^m_R := \mathrm{Proj}(R[t_0, \dots, t_m])$. Let $\mathcal{O}_X(1)$ be the restriction to X of the tautological ample sheaf of $\mathbf{P}^m_R$. Then there is an integer $d > 0$ such that $H^1(\mathbf{P}^m_R, \mathcal{I}_{X|\mathbf{P}^m_R}(d)) = 0$ and the d-th twist $\mathcal{I}_{C_i}(d)$ of the sheaf of ideals of C_i in X is globally generated for every $i = 1, \dots, n$.

Let $\xi_{n+1}, \dots, \xi_{n+r}$ denote the associated points of X_η. Then $\mathcal{I}_{C_i}$ is invertible at ξ_j for each $j = 1, \dots, n+r$. Indeed, this is clearly so if $j \neq i$ because $\xi_j \notin C_i$. On the other hand, $(\mathcal{I}_{C_i})_{\xi_i}$ is the ideal of $\mathcal{O}_{X,\xi_i}$ generated by π, which is a nonzero-divisor because f is flat, whence $\mathcal{I}_{C_i}$ is also invertible at ξ_i.

Since $\mathcal{I}_{C_i}(d)$ is globally generated, and since $H^1(\mathbf{P}^m_R, \mathcal{I}_{X|\mathbf{P}^m_R}(d)) = 0$, for each $j = 1, \dots, n+r$ there is a degree-d homogeneous polynomial $P_j \in R[t_0, \dots, t_m]$ generating $\mathcal{I}_{C_i}(d)$ at ξ_j. Since k is infinite, a sufficiently general linear combination $Q_i := \sum_j c_j P_j$ with $c_j \in k$ generates $\mathcal{I}_{C_i}(d)$ at ξ_j for every $j = 1, \dots, n+r$. Let $G_i \subseteq X$ be the subscheme cut out by $Q_i = 0$. Since G_i does not vanish on ξ_j for any $j = n+1, \dots, n+r$, the subscheme G_i is a Cartier divisor. It is indeed the Cartier divisor required by the first statement.

As for the second statement, as X_s is reduced, for each $j = 1, \dots, n$ the point ξ_j lies on the nonsingular locus of X_s, and hence on the nonsingular locus of X. So the local ring $\mathcal{O}_{X,\xi_j}$ is a discrete valuation ring.

For each $j = 1, \dots, n$, consider the ideal of D at ξ_j. If it were zero, then D would contain any irreducible closed subscheme of X containing ξ_j. However, among those there is at least one irreducible component of X whose generic point lies over η by flatness. Thus D would contain an irreducible component of X_η, contradicting the hypothesis that $D \cap X_\eta$ is a Cartier divisor. So the ideal of D at ξ_j is nonzero. Since $\mathcal{O}_{X,\xi_j}$ is a discrete valuation ring, this ideal is thus a power p_j of the maximal ideal. Let $G_1, \dots, G_n$ be the Cartier divisors of X claimed in the first statement. Set

$$E_i := \sum_{p_j < p_i} (p_i - p_j) G_j.$$

Then E_i is an effective Cartier divisor of X with $E_i \not\ni \xi_i$. Also, $D + E_i \supseteq p_i X_s$. Set $F_i := D + E_i - p_i X_s$. Then F_i is a subscheme of X with $F_i \not\ni \xi_i$. $\square$

5. Examples

Example 5.1 (*See [4], Ex. 11.3.2, p. 203, and the references listed there*). Let F, A_1, A_2, G_1 and G_2 be forms defining hypersurfaces. Assume FA_i and G_j have the same degree, for $i = 1, 2$ and $j = 1, 2$. Assume as well that

$$\gcd(FA_2, A_1) = 1 \quad \text{and} \quad \gcd(A_1G_2 - A_2G_1, F) = 1. \tag{5.1.1}$$

Consider the pencils $FA_i + tG_i = 0$ for $i = 1, 2$, and their intersection. The hypotheses (5.1.1) imply that the intersection is proper for a general t. Indeed, if the intersection were not proper, then there would be forms L_0 and L_1 of the same degree such that the polynomial $L_0 + TL_1$ divides $FA_1 + TG_1$ and $FA_2 + TG_2$. But then L_0 would divide FA_1, FA_2 and $A_1G_2 - A_2G_1$.

Let W denote the limit of the intersection of the curves given by $FA_1 + tG_1$ and $FA_2 + tG_2$ as t goes to 0. The intersection of the hypersurfaces $FA_1 = 0$ and $FA_2 = 0$ is not proper, and thus does not reflect W well. However, $FA_2 = 0$ cuts a Cartier divisor on $A_1 = 0$. In addition,

$$A_1(FA_2 + tG_2) = A_2(FA_1 + tG_1) + t(A_1G_2 - A_2G_1),$$

and $A_1G_2 - A_2G_1 = 0$ cuts a Cartier divisor on $F = 0$. Thus, by Theorem 4.1,

$$\begin{aligned} [W] &= [FA_2, A_1 = 0] + [A_1G_2 - A_2G_1, F = 0] - [A_1, F = 0] \\ &= [A_2, A_1 = 0] + [A_1G_2 - A_2G_1, F = 0]. \end{aligned}$$

Example 5.2 (*See [4], Ex. 11.3.3, p. 203*). Consider the families of plane curves parameterized by t:

$$x^2y - tz^3 = 0 \quad \text{and} \quad (x - t^2y)(y^2 - t^2x^2) = 0.$$

It is easy to compute the intersection of the above curves for general t, as the second curve is a union of lines. Letting β be a primitive cubic root of unity, we see that the intersection is reduced and consists of the nine points

$$(t^2 : 1 : t\beta^j), \quad (1 : t : \beta^j) \quad \text{and} \quad (1 : -t : -\beta^j) \text{ for } j = 0, 1, 2.$$

As t goes to 0, the $(t^2 : 1 : t\beta^j)$ approach $(0 : 1 : 0)$, while the remainder approach the six points $(1 : 0 : \pm\beta^j)$.

To compute these limits using Theorem 4.1, we first use Proposition 3.2 to reduce the problem to that of computing the limits of the Cartier divisors cut on $x^2y - tz^3 = 0$ by $x - t^2y = 0$ and by the lines $y \pm tx$. Call the first limit D and the last two limits $D_\pm$.

We will actually use Theorem 4.1 to compute $2[D]$, and then use Proposition 3.2 to get $[D]$. First, $x^2 = 0$ cuts a Cartier divisor on $y = 0$. Also,

$$y(x - t^2y)^2 \equiv (x^2y - tz^3) + tz^3 \mod t^2,$$

and $z^3 = 0$ cuts a Cartier divisor on $x^2 = 0$. Thus, by Theorem 4.1,

$$2[D] = [x^2, y = 0] + [z^3, x^2 = 0] - [y, x^2 = 0] = 6[z, x = 0].$$

So $[D] = 3[(0 : 1 : 0)]$.

As for $D_\pm$, first, $y = 0$ cuts out a Cartier divisor on $x^2 = 0$. Also,

$$x^2(y \pm tx) = (x^2y - tz^3) + t(z^3 \pm x^3),$$

and $z^3 \pm x^3$ cuts a Cartier divisor on $y = 0$. Thus, by Theorem 4.1,

$$[D_\pm] = [y, x^2 = 0] + [z^3 \pm x^3, y = 0] - [x^2, y = 0] = \sum_{j=0}^{2}[(1 : 0 : \mp\beta^j)].$$

Example 5.3. If a plane curve is smooth, its flexes are cut out by the Hessian, the determinant of the symmetric matrix of second-order partial derivatives of the form defining the curve. However, if the curve has a linear or a multiple component, the Hessian vanishes completely on that component. What are the possible limits of flexes on the curve if the curve has such components?

The above question, considered in [2], p. 151, will also be considered in more detail in [3]. Here we will just consider the simple case where the curve is reduced with just two components, and just one of them is linear, and where the curve is deformed in first order along a general direction.

Let k be any algebraically closed field with characteristic zero. For each $P \in k[[t]][x, y, z]$, define the derivation $D_P := \partial_y(P)\partial_x - \partial_x(P)\partial_y$, where ∂_x, ∂_y and ∂_z are the canonical partial $k[[t]]$-derivations of $k[[t]][x, y, z]$. Notice that $D_P(P) = 0$. For short, let $P_x := \partial_x(P)$, $P_y := \partial_y(P)$ and $P_z := \partial_z(P)$. Define the Hessian determinant $H(P)$ and the Wronskian determinant $W(P)$:

$$H(P) := \begin{vmatrix} P_{x,x} & P_{x,y} & P_{x,z} \\ P_{y,x} & P_{y,y} & P_{y,z} \\ P_{z,x} & P_{z,y} & P_{z,z} \end{vmatrix} \quad \text{and} \quad W(P) := \begin{vmatrix} x & y & z \\ D_P(x) & D_P(y) & D_P(z) \\ D_P^2(x) & D_P^2(y) & D_P^2(z) \end{vmatrix}.$$

If P is homogeneous of degree p, it follows from applying the Euler formula twice that

$$z^3H(P) \equiv (p-1)^2W(P) \mod P. \tag{5.3.1}$$

Let

$$F := xG - F_1t \in k[[t]][x, y, z],$$

where G and F_1 are nonzero forms of degrees $d - 1$ and d, respectively, for an integer $d \geq 3$. Assume G is irreducible and $\gcd(xz, G) = 1$. Then $D_F \equiv xD_G \mod (G, t)$, and hence $D_F(G) \equiv 0 \mod (G, t)$. So, using the multilinearity of the determinant, the product rule for derivations, and (5.3.1) for $P := G$, we get

$$(d-1)^2W(F) \equiv (xz)^3H(G) \mod (G, t). \tag{5.3.2}$$

Since G is irreducible and nonlinear, $H(G)$ cuts a divisor on the curve defined by G. Let R_G denote the 0-cycle of $\mathbb{P}^2$ associated with this divisor. Since G is neither a multiple of x nor of z, it follows from (5.3.2) that also $W(F)_0$ cuts a divisor on the curve given by G, where $W(F)_0 := W(F)|_{t=0}$.

Since $D_F(F) = 0$, using the multilinearity of the determinant and the product rule for derivations, we get

$$G^3W(F) \equiv tW' \mod F, \tag{5.3.3}$$

where W' is the Wronskian determinant:

$$W' := \begin{vmatrix} F_1 & Gy & Gz \\ D_F(F_1) & D_F(Gy) & D_F(Gz) \\ D_F^2(F_1) & D_F^2(Gy) & D_F^2(Gz) \end{vmatrix}.$$

Now, $D_F \equiv -G\partial_y \mod (x, t)$, and so $D_F(x) \equiv 0 \mod (x, t)$. Thus, using the multilinearity of the determinant and the product rule for derivations, we get

$$W' \equiv -g^3w \mod (x, t), \tag{5.3.4}$$

where $g := G(0, y, z)$ and w is the Wronksian determinant:

$$w := \begin{vmatrix} f' & g' & g'' \\ f'_y & g'_y & g''_y \\ f'_{y,y} & g'_{y,y} & g''_{y,y} \end{vmatrix},$$

with $f' := F_1(0, y, z)$, $g' := gy$ and $g'' := gz$. If F_1 is general enough – more precisely, if $F_1 \notin (x, G)$ – then $w \neq 0$, and hence also $W' \not\equiv 0 \bmod (x, t)$.

Now, since f', g' and g'' have degree d, applying the Euler formula three times, we get

$$(d-1)^2 dw = z^3 h, \tag{5.3.5}$$

where h is the "Hessian" determinant:

$$h := \begin{vmatrix} f'_{z,z} & g'_{z,z} & g''_{z,z} \\ f'_{y,z} & g'_{y,z} & g''_{y,z} \\ f'_{y,y} & g'_{y,y} & g''_{y,y} \end{vmatrix}.$$

Let R_x denote the 0-cycle of $\mathbb{P}^2$ of the scheme given by $h = x = 0$.

For any two coprime forms P_1 and P_2 of $k[x, y, z]$, denote by $[P_1 \cdot P_2]$ the 0-cycle of $\mathbb{P}^2$ of the scheme given by $P_1 = P_2 = 0$.

Assume F_1 is general. Let $S := \mathrm{Spec}(k[[t]])$ and $X \subseteq \mathbf{P}^2_S$ be the subscheme cut out by F. Let $f : X \to S$ denote the structure map. Let $E \subseteq X$ be the subscheme cut out by $W(F)$. Since F_1 is general, the general fiber X_η of f is smooth, and thus, by (5.3.1) applied to $P := F$, the subscheme E cuts X_η in its divisor of flexes plus 3 times the hyperplane section given by $z = 0$. So, E is a Cartier divisor of X. And, by Proposition 3.2, $[\lim E] = 3[z \cdot (xG)] + R$, where R is the 0-cycle of the schematic boundary of the divisor of flexes of X_η.

Using (5.3.3) and Theorem 4.1,

$$[\lim E] = [W(F)_0 \cdot G] + [W'_0 \cdot x] - 3[G \cdot x],$$

where $W'_0 := W'|_{t=0}$. In addition, by (5.3.2), (5.3.4) and (5.3.5),

$$[W(F)_0 \cdot G] = 3[x \cdot G] + 3[z \cdot G] + R_G,$$
$$[W'_0 \cdot x] = 3[G \cdot x] + 3[z \cdot x] + R_x.$$

Thus,

$$R = R_G + R_x + 3(G \cdot x).$$

In words, the limits of flexes are the flexes on the curve given by G plus 3 times the points of intersection of that curve with the line given by x plus a divisor R_x on the line, given by h. If the curves defined by F_1 and G intersect transversally, then it can be seen that R_x is the ramification divisor of the linear system cut out on the line by the degree-d curves passing through the intersection points. So, the base points of the pencil determine R_x.

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